

The Machian contribution of the Universe to geodetic precession, frame dragging and gravitational clock effect

P. Christillin
Dipartimento di Fisica,
Università di Pisa
I.N.F.N. Sezione di Pisa

and

L. Barattini
Università di Pisa

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Abstract

Gravitomagnetism resulting from SR has been applied to geodetic precession and frame dragging. The substantial contribution of the “fictitious” Coriolis force, due to the relative rotation of the rest of the Universe in the non inertial frame of the free falling but rotating satellite, has to be taken into account, giving another quantitative confirmation of Mach’s arguments and of the black hole nature of our Universe. Also the gravitational clock effect has an elementary prediction in the present post Newtonian formulation.

1 Introduction

Recently a set of “Heaviside” vector equations for gravity has been *derived* from special relativity and shown to predict in simple terms the quadrupole gravitational radiation [1].

They are effective in the sense that they are valid up to $O(v^2/c^2)$, self energy effects contributing only to a higher order in this expansion parameter.

They can thus provide a parameter free prediction for the weak gravitomagnetic field produced by the rotation and revolution of the earth on an orbiting satellite (Lageos [2] and Gravity Probe B [3]), this stationary situation representing just a particular case of the time dependent equations.

2 The vector equations

The vector equations for gravitation are the following :

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \quad (1)$$

$$\nabla \cdot \mathbf{h} = 0 \quad (2)$$

in the first \mathbf{g} representing the “ordinary” Newtonian field, while the second for the gravitomagnetic field \mathbf{h} based on the *assumption* (a fortiori even more reasonable than in electromagnetism) of the non existence of a gravitomagnetic charge.

These two are accompanied by the time dependent ones :

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{h}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{h} = -\frac{4\pi G}{c^2} 2\mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{g}}{\partial t} \quad (4)$$

The gravitomagnetic equation differs by the corresponding Maxwell one by the factor of 2 in front of the ordinary mass current density \mathbf{j} , required by special relativity.

Thus a post Newtonian formulation of gravitation has necessarily to embody a short distance repulsion from self energy effects (which modifies Newton’s law) and velocity dependent, possibly repulsive terms, both effects, somewhat at variance with the standard picture, coming from elementary considerations.

The time dependent terms which are crucial in determining the wave equation, play no role here, since we will consider only stationary conditions and the gravitomagnetic field generated by mass currents (self energy effects having been shown to be irrelevant), so that

$$\nabla \times \mathbf{h} = -\frac{4\pi G}{c^2} 2\mathbf{j} \quad (5)$$

thus implying a Lorentz gravitomagnetic force

$$\mathbf{F} = m(\mathbf{g} + \mathbf{v} \times \mathbf{h}) \quad (6)$$

where m is the relativistic mass.

It is worth stressing that *Eq. (5) unambiguously determines the magnetic part of the Lorentz force Eq. (6)*, the product of \mathbf{v} and \mathbf{h} coming just from Lorentz transformations. This point will be commented upon at length later on.

Let us mention the extra constraint which additionally backs up the present considerations. The induction law in its integral formulation, for the case of constant \mathbf{h} and a varying circuit is in agreement with the Lorentz force *only* in the present form. This represents therefore a double confirmation of the present formulas.

3 Geodetic precession and frame dragging effects

Let us then come to the Lageos [2] and Gravity Probe B experiments [3]. As well known the latter measures the effects of the orbital motion of the earth (of mass M_E) around the satellite (geodetic precession) and of its rotation (frame dragging) on satellite mounted gyroscopes at an altitude of 642 km. In both cases the relevant parameter which determines the angular velocities of the gyroscopes (apart from the numerical coefficients which will be given in the following) is, as usual,

$$\frac{GM_E}{c^2 R} \simeq 10^{-9} \quad (7)$$

where $R \simeq 7000$ km. This, because of the preceding considerations about the successful effective vector formulation of gravity and the smallness of the effect, casts more than reasonable doubts as to whether these precessions should be unambiguously attributed to GR.

Thus one has for the gravitomagnetic field of a loop of radius R described by the earth at the origin (i.e. the place of the satellite in its reference frame, around which the earth revolves)

$$h_{orb} = 4 \frac{G\mu}{c^2 R^3} = 2 \frac{GM_E}{c^2 R} \omega_{orb} \quad (8)$$

with the straightforward dipole extension to any direction.

The so called geodetic precession is simply due to the the angular velocity of precession of the (gravito) magnetic moment $\boldsymbol{\mu}$ of the satellite gyroscope (of standard angular momentum $\boldsymbol{S} = mr^2 \omega_{orb} \boldsymbol{n} = 2\boldsymbol{\mu}$) in a gravitomagnetic field \boldsymbol{h} which is governed by the Newtonian equation

$$\frac{1}{2} \boldsymbol{S} \times \boldsymbol{h} = \frac{d\boldsymbol{S}}{dt} \quad (9)$$

This implies

$$\Omega_{geo} = h/2 \quad (10)$$

To this *spin orbit effect*, trivially governed by classical mechanism and SR (calculation of \boldsymbol{h}), one must add the Thomas (T) precession, again due solely to SR. An elementary derivation of the Thomas precession, in terms of proper time (the time on the satellite is not the time observed on the rotating earth which to a good approximation can be taken to be that of the fixed stars) can be found in [4], with the result $\Omega_T = -\frac{GM_E}{2Rc^2} \omega_{orb}$.

Notice that since the satellite is seen to precess, in its reference frame it recedes with respect to the fixed stars. Thus the total angular velocity of precession due to the earth revolution is

$$\frac{\Omega_{geo}}{\omega_{orb}} = \frac{1}{2} \frac{GM_E}{c^2 R} \quad (11)$$

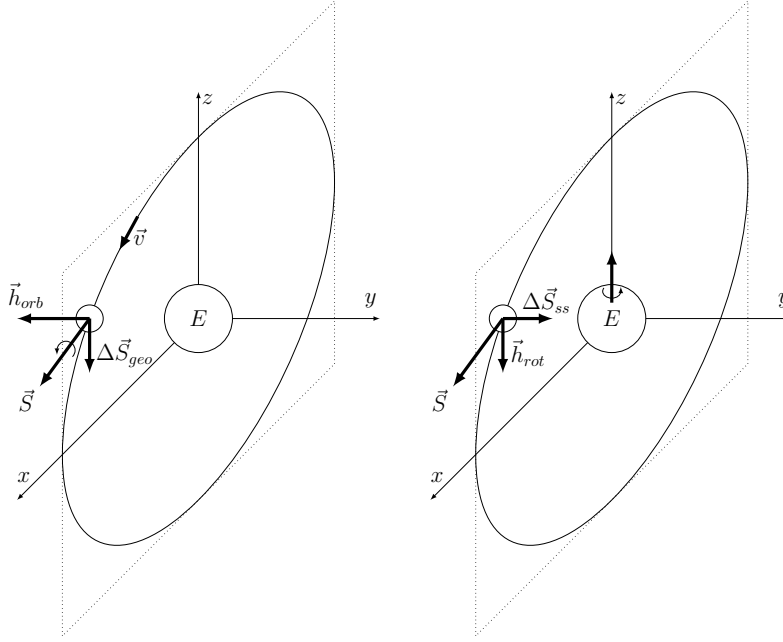


Figure 1: (Left) The satellite spin \vec{S} lies in the (x, z) plane described by its orbital motion around the earth. The gravitomagnetic loop described by the latter around \vec{S} , generates on it \vec{h}_{orb} perpendicular to the plane which makes the gyroscope to precess, even with the inclusion of the Thomas recession. (Right) The spinning earth generates an additional gyromagnetic field \vec{h}_{rot} on the spinning gyroscope \vec{S} , this time in the orbital plane. The gravitational spin-spin interaction makes the plane rotate around z (frame dragging). Both effects are predicted by SR in a flat Euclidean space

i.e. a relative effect determined by the (weak) gravitational field strength parameter, as illustrated in Fig.(1) Left.

The present result Eq. (10) differs by a factor of 2 from the GR calculations by Schiff [8].

Also numerous NR reductions of GR for the weak field and low velocity case have been recently appeared [10, 11, 12, 13, 14]. Apart from their problems with wave propagation, critically commented upon in [1], they seem to confirm Schiff's result only by introducing an extra factor of 2 in the Lorentz force Eq. (6).

As underlined before, this is forbidden just by SR transformations which connect Eq.s (1), (5), (6), and (3).

Let us then consider the smaller effect due to the magnetic field created by the earth rotation around its axis: the *spin-spin effect* usually dubbed “frame dragging effect”. It goes without saying how special relativity is again all one

needs. The gravitomagnetic dipole moment of a mass element dm of a rotating body is $d\mu = \frac{1}{2}\omega r^2 dm$, so that the gravitomagnetic field of the spinning earth at the gyroscope in an arbitrary direction reads

$$\mathbf{h}_{rot} = \frac{GI_E}{c^2 R^3} [(3\boldsymbol{\omega}_{rot} \cdot \mathbf{n})\mathbf{n} - \boldsymbol{\omega}_{rot}] \quad (12)$$

\mathbf{n} standing for the unit vector along \mathbf{R} . Now the previous torque equation (9) obtains so that one immediately gets the resulting precession in terms of the angular velocity of rotation of the earth ω_{rot}

$$\Omega_{spin-spin} = h_{rot}/2 \quad (13)$$

Its direction, at right angles with the geodetic precession, is shown in Fig. (1) Right. This time again a direct (i.e. without the intermediary of the Thomas precession) factor of 1/2 results from the comparison with the existing literature.

4 Discussion

As mentioned our predictions differ from the quoted measurements by a factor of 1/2. Some extensive comments are then in order.

The GR based predictions have been confirmed to a different degree of accuracy by the Lageos [2] and Gravity Probe B experiments [3] and will be further scrutinized by the proposed multi-ring-laser underground experiment [5].

The situation appears hence somewhat contradictory. NR reductions of GR equations give a vector formulation which (while confirming the soundness of the present approach) is in agreement with GR Schiff's results only at the price of a wrong Lorentz force !

Therefore, granting the correctness of the experimental results and apart from it, the basic question we have to address is: is the doubly rotatory motion of the satellite gyroscope S determined only by the earth motion ?

It is indeed clear that the gyroscopes are just an up to date version of Newton's bucket. Therefore, if in line with Mach's thinking, we do not believe in absolute motion we have to ascertain the role of the relative motion of rest of the Universe. This time quantitatively, since the presumed sole (and dominant) contribution of the earth has a quantitative estimate.

The point is that the free fall satellite frame is an inertial one so long as it does not rotate. Once it does, due to the earth effect, it no longer is. We must therefore introduce Coriolis forces or the effect of the rest of the Universe.

$$\mathbf{F}_{Cor} = 2m \mathbf{v} \times \boldsymbol{\omega} \quad (14)$$

from which [17]

$$\mathbf{M}_{Cor} = \mathbf{S} \times \boldsymbol{\omega} \quad (15)$$

where ω refers respectively (and separately) to each of the two rotations induced by the movement of the earth. Thus we have to add to Eq. (9) this extra contribution, obtaining a total rotation

$$\Omega = h \quad (16)$$

instead of the previous $\Omega = h/2$.

Thus our prediction of the spin spin precession is simply doubled whereas for the geodetic precession the doubling of Ω combined with the unaffected Thomas precession yields a final factor of 3/2 for Ω_{geo} , this time again in accord with Schiff's result.

The previous result provides a deeper understanding of the (non) equivalence principle: *the fact that forces are locally eliminated in the free falling frame (no tide effects), does not imply the same for the moments !* [16]

We are then led to revisit Sciama's conjecture [18] who has greatly emphasized the similarity between the previous gravitomagnetic force and the "fictitious" Coriolis force experienced in a rotating frame, stressing *the connection between angular velocity of rotation and corresponding magnetic field. The proportionality coefficient being simply given by the ubiquitous factor $GM/(c^2 R)$!*

Indeed as an extension to the (rest of the) Universe of the previous expression for the gravitomagnetic field of a mass m it follows

$$\mathbf{F}_{GM} = m \mathbf{v} \times \left(\frac{2GM}{c^2 R} \boldsymbol{\omega} \right) = 2mv \frac{GM}{c^2 R} \boldsymbol{\omega} \mathbf{n} \quad (17)$$

the suffix GM standing for gravitomagnetic.

Thus if

$$\frac{GM_U}{c^2 R_U} = 1 \quad (18)$$

then

$$\mathbf{F}_{Cor} = \mathbf{F}_{GM} \quad (19)$$

The essential point in this argument is that in the relative rotation of the satellite with respect to the Universe, the magnetic field generated by distant layers of matter goes as $1/R$ i.e. the same behaviour of radiation, rather than the usual $1/R^2$ of Newton forces. Therefore a relative more important role even of distant stars is a matter of fact.

In favour of the estimate/Ansatz of Eq. (18) there is a lot of circumstantial evidence as well as speculations [7]. In particular it is necessary to account for the precession of the Foucault pendulum as determined along the present lines by the rotating matter of the Universe.

5 The gravitational clock effect

With inclusion of the gravitomagnetic force, the two body gravitational equation of motion thus reads

$$m\omega^2 R = \frac{GMm}{R^2} + mvh \quad (20)$$

where the sign of the last term, depending on the relative orientation of the velocity of the mass m orbiting the spinning mass M will be detailed at the

end. For the case of two opposite orbiting satellites in the equatorial plane, the previous equation of $h = h_{rot}$ in a simplified form holds

$$\mathbf{h}_{rot} = -\frac{GI_E}{c^2 R^3} \boldsymbol{\omega}_{rot} \quad (21)$$

Here the Thomas precession which affects in the same way both satellites has been omitted.

Thus in terms of the angular momentum S of the spinning mass M (the earth), of the Keplerian angular velocity $\omega_K = \sqrt{GM/R^3}$ and of the post Keplerian correction τ

$$\tau = S/Mc^2 \quad (22)$$

one has

$$\omega^2 = \omega_K^2 + \omega_K^2 \omega \tau \quad (23)$$

The admissible root is

$$\omega \simeq \omega_K + \frac{1}{2} \omega_K^2 \tau \quad (24)$$

where terms of higher order in τ have been neglected.

Thus

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_K + \frac{1}{2} \omega_K^2 \tau} \simeq T_K - 2\pi \frac{\tau}{2} \quad (25)$$

As in the preceding case the result differs by the GR prediction by a factor of $1/2$!

But *exactly as in the preceding case* the effect of the Coriolis force i.e. of the “rotating fixed stars” has to be taken into account.

In other words the complete projected equation of motion reads

$$m\omega^2 r = \frac{GMm}{r^2} + mvh + 2mv \omega_{rot} \quad (26)$$

This trivially yields along the lines of the preceding paragraph a Coriolis contribution

$$T_C = -2\pi \frac{\tau}{2} \quad (27)$$

with an ensuing post Keplerian correction

$$\Delta T_{pK} = \pm 2\pi\tau \quad (28)$$

which agrees with the GR result [15].

The plus sign applies for the same sense of rotation of the satellite and the earth, whereas the minus (smaller period) for antirotation.

We are therefore in the presence, in principle, of an additional test of gravitomagnetism where SR is enough to predict the results of GR.

6 Conclusions

Gravitomagnetism resulting from SR yields a set of parameter free vector equations which provide an effective theory of gravitation.

They have been shown to predict in elementary terms the quadrupole gravitational radiation in a flat Minkowski space.

In this work the particular case of stationary currents has been considered and applied to geodetic precession, frame dragging and the gravitational clock effect.

It has been shown, contrary to naive expectations, that the orbital and spin rotation of the earth do not account for the experimental results, the contribution of the rotating Universe on the non inertial frame of the satellite being of the same order of magnitude of the earth's.

The following comments are inevitable :

- in this case SR is all we need to get the GR results
- the Machian picture gets a piece of support and the role of counterrotating fixed stars is paramount
- GR, in spite of its claims of generality, assumes a privileged reference frame [19] ! Empirically, this system coincides with the average system of the fixed stars, however, this correspondence appears incidentally, since the presence of the distant masses did in no way enter the calculation.

Thus Mach's thinking enters quite rightly and quantitatively our picture of the Universe through the prediction of the "fictitious" Coriolis force in a post Newtonian language which, in our opinion, has the advantage, besides its simpler formulation, of making such a connection plain !

Therefore it is really rewarding to have such a deep link between local and global properties of the Universe.

We must unescapably accept the existence of a privileged frame of reference: the microwave background radiation, to a very good degree of accuracy, takes the place and confirms the "fixed stars system"!

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- [16] This effect has not been considered so far. Indeed Schiff explicitly writes that the satellite is "in free fall". The precession takes place with respect to the inertial frame, which is generally believed to be defined by the distant extragalactic nebulae, the so called "fixed stars".
- [17] This apparently contradictory result is due to the elementary but sometimes overlooked fact that whereas $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\boldsymbol{\omega} = \frac{1}{2}\mathbf{r} \times \mathbf{v}$.
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 "Ironically, though general relativity was intended to be based on relational concepts, contrary to its name it still contains absolute elements . This is already expressed in the calculation of the advance of Mercury's perihelion, which is referred to a coordinate system .."